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Abstract: We compute a naturally defined measure of the size of the nef cone of a Del Pezzo surface. The resulting number appears in a conjecture of Manin on the asymptotic behavior of the number of rational points of bounded height on the surface. The nef cone volume of a Del Pezzo surface Y with (−2)-curves defined over an algebraically closed field is equal to the nef cone volume of a smooth Del Pezzo surface of the same degree divided by the order of the Weyl group of a simply-laced root system associated to the configuration of (−2)-curves on Y . When Y is defined over an arbitrary perfect field, a similar result holds, except that the associated root system is no longer necessarily simply-laced.

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A NEF CONE VOLUME FOR GENERALIZED DEL PEZZO SURFACES

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Abstract. For a smooth projective algebraic variety X, $\alpha(X)$ is a positive real number measuring the size of the dual to the cone of effective divisors on X. A conjecture of Manin predicts an asymptotic expression for the number of rational points of bounded height on X, in which the constant $\alpha(X)$ appears. Values of $\alpha(X)$ were found by Derenthal in [9] for split Del Pezzo surfaces, and also for split generalized Del Pezzo surfaces using a computer calculation. We extend these results both via an inductive method and by using the action of the Weyl group on the nef cone. Finally, we compute $\alpha(X)$ for non-split ordinary and generalized Del Pezzo surfaces of degree ≥ 5.

1. Introduction

We compute a naturally defined volume of a polytope in the nef cone of a generalized Del Pezzo surface Y. When Y is defined over a number field, this volume plays a role in a conjecture of Manin on the number of rational points of bounded height on Y.

Recall that a Del Pezzo surface—which we will call an ordinary Del Pezzo surface to distinguish from the generalized Del Pezzo surfaces we consider in this paper—is a smooth projective rational surface X on which the anticanonical class $-K_X$ is ample. A **generalized Del Pezzo surface** is a smooth projective rational surface Y on which $-K_Y$ is big and nef. If Y is defined over an algebraically closed field, Y is one of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, the Hirzebruch surface $\mathbb{F}_2$, or $\mathbb{P}^2$ blown up at $1 \leq r \leq 8$ points in "almost general position" [8]. To blow up r points on $\mathbb{P}^2$ in almost general position is to construct a sequence of morphisms

$$Y = Y_r \to Y_{r-1} \to \cdots \to Y_1 \to Y_0 = \mathbb{P}^2,$$

where each map $Y_{i+1} \to Y_i$ is the blow-up of Y_i at a point $P_i \in Y_i$ not lying on any curves of self-intersection -2 on Y_i. The resulting surface Y may have (-2)-curves, but no curves of self-intersection -3 or lower. The **degree** of Y is $d = (-K_Y)^2$. The degree is always in the range $1 \leq d \leq 9$, and when Y is the blowup of r points in almost general position, $d = 9 - r$. For Y of degree $d \geq 3$, the anticanonical morphism defined by the linear series $|-K_Y|$ embeds Y in a projective space of dimension d.

Over an algebraically closed field, a generalized Del Pezzo surface is ordinary if and only if it contains no effective curves with self-intersection -2. If the field of definition K is not

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TABLE 1. Values of α for ordinary Del Pezzo surfaces

d	8	7	6	5	4	3	2	1
r	1	2	3	4	5	6	7	8
$\alpha(X_d)$	1/6	1/24	1/72	1/144	1/180	1/120	1/30	1

algebraically closed, we say that a generalized Del Pezzo surface Y is **split** if the Galois action on the Picard group of $\overline{Y} = Y \times_K \overline{K}$ is trivial.

For generalized Del Pezzo surfaces, the constant $\alpha(Y)$ is equal to the volume of the polytope obtained by slicing the nef cone of Y with the hyperplane of divisor classes whose intersection with the anticanonical class is 1. We give more details of this definition in section 2.

An investigation of $\alpha(Y)$ for generalized Del Pezzo surfaces was begun by U. Derenthal in [9]. He proved the following result:

Theorem 1.1. *Let X_d be a split ordinary Del Pezzo surface of degree $d \leq 6$. Let N_d denote the number of (-1)-curves on X_d. Then*

$$\alpha(X_d) = \frac{N_d}{d(9-d)}\alpha(X_{d+1}).$$

Combining this with a simple calculation that shows $\alpha(X_7) = 1/24$, and a calculation by Peyre [16, Lemme 9.4.2] for $d = 8$, this theorem allows for an inductive calculation of the alpha constant of any split ordinary Del Pezzo surface. This calculation is summarized in Table 1.

We extend this result to split generalized Del Pezzo surfaces:

Theorem 1.2. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 7$. Let $\mathcal{I}_Y$ denote the set of irreducible (-1)-curves on Y. For each $E \in \mathcal{I}_Y$, let Y_E denote the split generalized Del Pezzo surface of degree $d+1$ obtained by blowing down E. Then*

$$\alpha(Y) = \sum_{E \in \mathcal{I}_Y} \frac{1}{d(9-d)}\alpha(Y_E).$$

Derenthal also computed α for split generalized Del Pezzo surfaces of degree $d \geq 3$ by using the computer program `POLYMAKE` to find a triangulation of the nef cone. His numerical data led us to formulate the following.

Theorem 1.3. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 6$ and let X be a split ordinary Del Pezzo surface of the same degree. Then*

$$\alpha(Y) = \frac{1}{\#W(R_Y)}\alpha(X),$$

where $W(R_Y)$ is the Weyl group of the root system R_Y consisting of the effective and anti-effective (-2)-classes on Y.

When combined with Theorem 1.1 the computation of α for an arbitrary split generalized Del Pezzo surface of any degree is reduced to a determination of the effective (-2)-curves on the surface.

The primary motivation for our study of α is its appearance in Manin's conjecture on the number of rational points of bounded height on Fano varieties defined over number fields. Although the conjecture is now known not to hold for all Fano varieties [4], it has been verified in a large number of cases, including some varieties for which $-K_X$ is big but not ample. Let X be a smooth projective variety defined over a number field for which the anticanonical class $-K_X$ is big and assume that the set $X(K)$ of rational points is Zariski dense. Equip $X(K)$ with an anticanonical height function H (consult [13] for information on height functions) and for any constructible set $U \subset X$ let

$$\mathcal{N}_U(B) := \#\{P \in U(K) : H(P) \leq B\}.$$

The original formulation of the Manin conjecture [1] is the existence of a Zariski open set $U \subset X$ such that for any open set $V \subset U$

$$\mathcal{N}_V(B) = c(X)B(\log B)^{\rho-1}(1 + o(1)),$$

where $\rho = \text{rank } N^1(X)$ and $c(X)$ is an unspecified constant. Peyre [16] later refined the conjecture by predicting that

$$c(X) = \alpha(X)\beta(X)\tau(X),$$

where $\alpha(X) \in \mathbb{Q}$ is the constant of interest in this paper, $\beta(X) \in \mathbb{N}$ is a cohomological invariant of the Galois action on $N^1(X)$, and $\tau(X) \in \mathbb{R}$ is a volume of adelic points on X. (Actually, the factor $\beta(X)$ was not considered by Peyre, but later discovered by Batyrev and Tschinkel [3] in their study of Manin's conjecture for toric varieties.) While $\tau(X)$ depends on the choice of the height function H, the invariants $\alpha(X)$ and $\beta(X)$ do not.

The remainder of the paper is organized as follows. In Section 2, we recall the precise definition of α for varieties which have a big anticanonical class. In Section 3, we prove that the effective cone of a generalized Del Pezzo surface of degree $d \leq 7$ is generated by its (-1)- and (-2)-curves. This is central to both of our methods of computing α.

A proof of Theorem 1.2 is given in Section 4. The key step identifies the nef cone of a blowdown of a (-1)-curve E on a generalized Del Pezzo surface Y of degree $d \leq 7$ as the intersection of the nef cone of Y with the orthogonal complement of E. In Section 5 we prove Theorem 1.3 after first recalling the basic facts about Weyl groups used in the proof. In Section 6, we consider non-split generalized Del Pezzo surfaces and provide some data for the high degree cases.

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2. Definition of α

Here we state the definition of $\alpha(X)$, first introduced by Peyre in [16].

Let X be a smooth complete variety for which $-K_X$ is big. Numerical equivalence classes of divisors on X, $N^1(X)$, form a finitely-generated torsion-free abelian group whose dual group $N_1(X)$ consists of numerical equivalence classes of 1-cycles on X. Let $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_1(X)_{\mathbb{R}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ be the associated Euclidean spaces. Inside $N^1(X)_{\mathbb{R}}$ lies the effective cone $\mathrm{Eff}^1(X)$ spanned by the classes of effective divisors, which gives rise to the dual cone

$$\mathrm{Eff}^1(X)^\vee := \{C \in N_1(X)_{\mathbb{R}} : \langle D, C\rangle \geq 0 \ \forall D \in \mathrm{Eff}^1(X)\}.$$

This is the "movable cone" [5], [14, §11.4.C]. Note that when X is a surface, $\mathrm{Eff}^1(X)^\vee$ is simply the nef cone of X, in which case we will often employ the simpler notation $\mathrm{Nef}(X)$.

Since the cone $\mathrm{Eff}^1(X)^\vee$ has infinite volume in $N_1(X)_{\mathbb{R}}$, a natural means of measuring its "size" is to truncate the cone in an (anti)canonical manner. To do this, let

$$\mathcal{H}_X := \{C \in N_1(X)_{\mathbb{R}} : \langle -K_X, C\rangle = 1\}.$$

Note that since $-K_X$ is big by hypothesis, $\mathcal{H}_X$ intersects each ray of $\mathrm{Eff}^1(X)^\vee$. We endow $N_1(X)_{\mathbb{R}}$ with Lebesgue measure ds normalized so that $N_1(X)$ has covolume 1, and we endow $\mathcal{H}_X$ with the induced Leray measure $d\mu$ with respect to the linear form $\langle -K_X, \cdot\rangle$. That is, letting l be the linear form $l(v) = \langle -K_X, v\rangle$, we have $ds = d\mu \wedge dl$. We construct the polyhedron

$$\mathcal{P}_X := \mathrm{Eff}^1(X)^\vee \cap \mathcal{H}_X$$

and define

$$\alpha(X) := \mathrm{Vol}(\mathcal{P}_X). \tag{1}$$

There are variants of this definition which differ only by a dimensional factor. Let $\rho = \dim N_1(X)_{\mathbb{R}}$ and

$$\mathcal{C}_X := \{C \in \mathrm{Eff}^1(X)^\vee : 0 \leq \langle -K_X, C\rangle \leq 1\}$$

be the convex hull of $\mathcal{P}_X$ and the origin. Then a simple slicing argument shows that

$$\alpha(X) = \frac{1}{\rho}\mathrm{Vol}(\mathcal{C}_{\mathcal{X}}).$$

Additionally,

$$\alpha(X) = \frac{1}{(\rho-1)!}\int \cdots \int_{\mathrm{Eff}^1(X)^\vee} \exp\left(-\langle -K_X, s\rangle\right) ds,$$

with the bigness of $-K_X$ insuring the convergence of the integral.

3. Effective Cones of Generalized Del Pezzo Surfaces

Let Y be a generalized Del Pezzo surface defined over a number field K.

Theorem 3.1. *If Y is split over K and has degree $d \leq 7$, the effective cone of Y is generated by the set of irreducible, effective (-1)- and (-2)-classes.*

Recall that the set of such classes is finite; see, for example, [15]. We begin by noting that this result holds in the case of ordinary Del Pezzo surfaces.

Proposition 3.2. *Let X be a split ordinary Del Pezzo surface of degree $d \leq 7$. Then the effective cone of X is generated by the (-1)-classes on X, all of which are irreducible and effective.*

Proof. This can be proved directly (see [11, Theorem V.4.11] for a proof when $r = 6$) or can be taken as an immediate consequence of the calculation of generators for the Cox ring given in [2]. □

Proposition 3.3. *If Y is split over K, every (-1)-class in $N^1(Y)$ is effective.*

Proof. This is an immediate consequence of Theorem 2(c) in Section III.7 of [8]. □

Lemma 3.4. *Let S be a surface and let $D_1, \dots, D_k$ be irreducible effective divisors on S. Let Γ denote the cone generated by $D_1, \dots, D_k$. Then the effective cone of S is equal to Γ if and only if $\Gamma^\vee \subset \Gamma$.*

The proof of Lemma 3.4 is a very modest generalization of the proof of Proposition 4.5 in [12]. We include the proof here for completeness.

Proof. If the effective cone of S is equal to Γ then it is a closed cone. The nef cone $\mathrm{Nef}(S) = \Gamma^\vee$ is contained in the closure of the effective cone, which is just Γ. Conversely, let D be the class of an effective divisor. Then we can write $D = D' + a_1 D_1 + \cdots + a_k D_k$ with $a_i \geq 0$ and the linear system of D' having no base components among the D_i. (Simply factor out any base components among the D_i in the linear system of D.) It is clear that D' is contained in $\Gamma^\vee$, and by hypothesis, D' is consequently contained in Γ. Hence the same is true of D and the lemma is proved. □

Remark 3.5. Let X be a split ordinary Del Pezzo surface and let Y be a split generalized Del Pezzo surface of the same degree $d \leq 7$. We describe an isomorphism of $N^1(X)$ and $N^1(Y)$ which preserves the intersection form and takes $-K_X$ to $-K_Y$.

Say X is the blowup of $\mathbb{P}^2$ at points $p_1, \dots, p_r \in \mathbb{P}^2$, $r = 9-d$, with blowdown $\pi_X : X \to \mathbb{P}^2$, and say Y is obtained by blowing up $\mathbb{P}^2$ at points $q_1, \dots, q_r$:

$$\pi_Y : Y = Y_r \to Y_{r-1} \to \cdots \to Y_1 \to Y_0 = \mathbb{P}^2$$

where $Y_{j+1} = \mathrm{Bl}_{q_j}(Y_j)$, $q_j \in Y_j$. Let $E_{X,j}$ be the exceptional divisor over p_j, and let $E_{Y,j}$ be the total transform in Y of the exceptional divisor over q_j. (That is, if $f_j : Y \to Y_j$, then $E_{Y,j} = f_j^{-1}(q_j)$, scheme-theoretically).

Then $\mathrm{Pic}(X)$ is the free abelian group on $L_X = \pi_X^* \mathcal{O}_{\mathbb{P}^2}(1)$, $E_{X,1}, \dots, E_{X,r}$. Similarly, $\mathrm{Pic}(Y)$ is the free abelian group on $L_Y = \pi_Y^* \mathcal{O}_{\mathbb{P}^2}(1)$, $E_{Y,1}, \dots, E_{Y,r}$. The intersection form on $\mathrm{Pic}(X)$ is given in this basis by the diagonal matrix $\mathrm{diag}(1, -1, -1, \dots, -1)$; the intersection form on $\mathrm{Pic}(Y)$ is given in this basis by the same matrix. We have $-K_X = 3L_X - \sum E_{X,j}$ and $-K_Y = 3L_Y - \sum E_{Y,j}$.

Therefore we have an isomorphism of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(Y)$ by identifying L_X with L_Y, $E_{X,1}$ with $E_{Y,1}$, etc. This identification preserves the intersection form and anticanonical divisor. (Note that this identification is far from unique; see [10].) It induces an identification of $N^1(X)$ and $N^1(Y)$, under which we have

(1) $\mathrm{Eff}^1(X) \subset \mathrm{Eff}^1(Y)$,
(2) $\mathrm{Nef}(X) \supset \mathrm{Nef}(Y)$.

Indeed, the inclusion of effective cones follows from Proposition 3.2 and Proposition 3.3. The inclusion of nef cones follows by duality.

Proof of Theorem 3.1. Let Γ be the cone generated by the irreducible, effective (-1)- and (-2)-curves on Y. To prove the theorem, it suffices by Lemma 3.4 to show that $\Gamma^\vee \subset \Gamma$. Let X be a split ordinary Del Pezzo surface of the same degree as Y. Identify $N^1(X)$ and $N^1(Y)$ as in Remark 3.5. By Proposition 3.2, we have $\mathrm{Eff}^1(X) \subset \Gamma$. It follows immediately that $\Gamma^\vee \subset \mathrm{Eff}^1(X)^\vee$. From Lemma 3.4 we have $\mathrm{Eff}^1(X)^\vee \subset \mathrm{Eff}^1(X)$. Thus $\Gamma^\vee \subset \Gamma$ and hence $\Gamma = \mathrm{Eff}^1(Y)$. ∎

Proposition 3.6. *Let Y be a generalized Del Pezzo surface, not necessarily split. Let $G_K = \mathrm{Gal}(\overline{K}/K)$. Then*

$$\mathrm{Eff}^1(Y) = \mathrm{Eff}^1(\overline{Y})^{G_K}$$

Proof. It is clear that $\mathrm{Eff}^1(Y) \subseteq \mathrm{Eff}^1(\overline{Y})^{G_K}$. To show the reverse inclusion, first note that if D is any effective divisor on $\mathrm{Eff}^1(\overline{Y})$, letting L be a finite Galois extension of K over which D is defined, then $\sum_{\sigma\in\mathrm{Gal}(L/K)} \sigma(D) \in \mathrm{Eff}^1(Y)$. The proof follows from the simple observation that for any $D \in \mathrm{Eff}^1(\overline{Y})^{G_K}$ that is defined over a Galois extension L/K,

$$D = \frac{1}{\#\,\mathrm{Gal}(L/K)} \sum_{\sigma\in\mathrm{Gal}(L/K)} \sigma(D).$$

∎

Corollary 3.7. *Retaining the notation of Proposition 3.6, the set of irreducible, effective (-1)- and (-2)-curves on $\overline{Y}$ is permuted by the action of G_K on $N^1(\overline{Y})$. A minimal set of generators for the effective cone of Y consists of, for each orbit of G_K on this set, the sum of the classes in that orbit.*

4. Inductive method

Lemma 4.1. *Let Y be a generalized Del Pezzo surface, and E an irreducible exceptional curve on Y. Let Y_E be the generalized Del Pezzo surface obtained by blowing down E. Then the intersection of the nef cone of Y with $E^\perp$ can be canonically identified with the nef cone of Y_E.*

Proof. Let $\pi : Y \to Y_E$ denote the blowdown map. We begin by noting that $N^1(Y) = \pi^*(N^1(Y_E)) \oplus \mathbb{Z}E$. Consequently we can identify $\mathrm{Eff}^1(Y_E)$ with $(\pi^*(\mathrm{Eff}^1(Y_E)), 0)$ in $N^1(Y)$, and by duality we can identify $\mathrm{Nef}(Y_E)$ with $(\pi^*(\mathrm{Nef}(Y_E)), 0)$ in $N_1(Y)$.

Assume first that $D \in \mathrm{Nef}(Y) \cap E^{\perp}$. We want to show $D \in (\pi^*(\mathrm{Nef}(Y_E)), 0)$. Since $D \in E^{\perp}$, it is immediate that $D \in \pi^*(N^1(Y_E))$, and consequently it suffices to show that $\langle \pi_* D, D' \rangle \geq 0$ for every D' of $\mathrm{Eff}^1(Y_E)$. Given such a D', we have

$$\pi^* D' = D'' + \langle D'', E \rangle E,$$

where D'' is the strict transform of D', an effective divisor on Y. The projection formula gives

$$\langle \pi_* D, D' \rangle = \langle D, \pi^* D' \rangle = \langle D, D'' \rangle + \langle D'', E \rangle \langle D, E \rangle \geq 0,$$

using $D \in \mathrm{Nef}(Y)$ to show that the first term $\langle D, D'' \rangle$ is non-negative and $D \in E^{\perp}$ to show that the second term is zero.

Conversely, assume that $D' \in \mathrm{Nef}(Y_E)$. We want to show $(\pi^* D', 0) = \pi^* D' \in \mathrm{Nef}(Y) \cap E^{\perp}$. We have $\langle \pi^* D', E \rangle = 0$, so $\pi^* D' \in E^{\perp}$. Let D be any irreducible, effective curve on Y. Then by the projection formula, $\langle \pi^* D', D \rangle = \langle D', \pi_* D \rangle$, and this is 0 if π contracts D or ≥ 0 otherwise. Therefore $\pi^* D' \in \mathrm{Nef}(Y)$. □

Theorem 4.2. *Let Y be a generalized Del Pezzo surface of degree $d \leq 7$ and let $\mathcal{I}_Y$ be the collection of irreducible (-1)-curves on Y. For each $E \in \mathcal{I}_Y$, let Y_E be the generalized Del Pezzo surface obtained by blowing down E. Then*

$$\alpha(Y) = \frac{1}{d(9-d)} \sum_{E \in \mathcal{I}_Y} \alpha(Y_E).$$

Proof. We follow the argument used in [9]. Let $\mathcal{E}'$ be the set of irreducible, effective (-2)-classes on Y and let $\mathcal{E} = \mathcal{I}_Y \cup \mathcal{E}'$. Then $\mathcal{E}$ is exactly the set of generators for $\mathrm{Eff}^1(Y)$ described in Theorem 3.1. Recall that the hyperplane $\mathcal{H}_Y$ is defined as

$$\mathcal{H}_Y = \{C \in N_1(Y)_{\mathbb{R}} : \langle -K_Y, C \rangle = 1\}.$$

The intersection $\mathrm{Nef}(Y) \cap \mathcal{H}_Y$ is a polytope with faces corresponding to $E \in \mathcal{E}$. For $E \in \mathcal{E}$, let $\mathcal{P}_E \subset \mathcal{H}_Y$ be the convex hull of the vector $\frac{1}{d}(-K_Y)$ and the face $\mathcal{P}_Y \cap E^{\perp}$. Then

$$\mathcal{P}_Y = \mathrm{Nef}(Y) \cap \mathcal{H}_Y = \bigcup_{E \in \mathcal{E}} \mathcal{P}_E.$$

The intersection of any two of the $\mathcal{P}_E$ has volume zero in $\mathcal{H}_Y$ because the intersection lies in a subspace of dimension strictly less than that of $\mathcal{H}_Y$. Therefore,

$$\alpha(Y) = \mathrm{Vol}(\mathcal{P}_Y) = \sum_{E \in \mathcal{E}} \mathrm{Vol}(\mathcal{P}_E).$$

For $E \in \mathcal{E}'$, $\langle K_Y, E \rangle = 0$ and hence $\frac{1}{d}(-K_Y) \in E^{\perp}$. Thus for $E \in \mathcal{E}'$, $\mathcal{P}_E$ lies in a hyperplane of dimension $\dim(\mathcal{H}_Y) - 1$, and so $\mathcal{P}_E$ has volume zero. We then have

$$\mathrm{Vol}(\mathcal{P}_Y) = \sum_{E \in \mathcal{I}_Y} \mathrm{Vol}(\mathcal{P}_E).$$

TABLE 2. Classification of root systems R_d

d	6	5	4	3	2	1
R_d	$A_1 \times A_2$	A_4	D_5	E_6	E_7	E_8

For $E \in \mathcal{I}_Y$, let $\pi_E : Y \to Y_E$ be the contraction. By Lemma 4.1 we have $\pi_E^* \mathcal{H}_{Y_E} = \mathcal{H}_Y \cap E^\perp$. This identifies the base of the cone $\mathcal{P}_E$ as $\mathcal{P}_Y \cap E^\perp = \pi_E^* \mathcal{P}_{Y_E}$. Thus $\mathcal{P}_E$ is a cone of dimension $9-d$ with height $\frac{1}{d}$ and base volume $\mathrm{Vol}(\mathcal{P}_{Y_E})$. Consequently,

$$\mathrm{Vol}(\mathcal{P}_E) = \frac{1}{d(9-d)} \mathrm{Vol}(\mathcal{P}_{Y_E}) = \frac{1}{d(9-d)} \alpha(Y_E).$$

Summing over $E \in \mathcal{I}_Y$ gives the desired result. $\square$

Remark 4.3. This generalization explains why Derenthal's result, Theorem 1.1, does not hold for $d = 7$. When one blows down a (-1)-curve on an ordinary Del Pezzo surface of degree d for $d \leq 7$ the result is an ordinary Del Pezzo surface of degree $d+1$. For $d \leq 6$, the resulting ordinary Del Pezzo surfaces all have the same effective cone. This is no longer true when $d = 7$, where the resulting ordinary Del Pezzo surface of degree 8 can either be the blow-up of $\mathbb{P}^2$ at one point or $\mathbb{P}^1 \times \mathbb{P}^1$. Indeed, letting X_d denote the blow-up of $\mathbb{P}^2$ at $r = 9-d$ points in general position, we have

$$\alpha(X_7) = \frac{1}{14}(2\alpha(X_8) + \alpha(\mathbb{P}^1 \times \mathbb{P}^1)) = \frac{1}{24}$$

since $\alpha(X_8) = 1/6$ and $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = 1/4$.

5. Weyl groups

Let Y be a generalized Del Pezzo surface of degree $d \leq 6$. The (-2)-classes in $N^1(Y)$ form a root system [15]. For the convenience of the reader we recall some of the basic facts about this root system.

A root is a class $\alpha \in N^1(Y)$ such that $\langle \alpha, \alpha \rangle = -2$ and $\langle -K_Y, \alpha \rangle = 0$. Let R_d be the set of all roots. This is a finite set. For $d \leq 6$, it is a root system in the hyperplane $(-K_X)^\perp$ [6, VI.1]. The well-known classification of this root system is shown in Table 2. (See, for example, [15, §23-25].)

For $\alpha \in R_d$, define $s_\alpha : N^1(Y)_\mathbb{R} \to N^1(Y)_\mathbb{R}$ to be the orthogonal reflection through α:

$$s_\alpha(x) = x - \frac{2\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle} = x + \langle \alpha, x \rangle \alpha$$

The group of orthogonal transformations generated by the s_α is the Weyl group $W_d = W(R_d)$. A wall in $N^1(Y)_\mathbb{R}$ is a hyperplane orthogonal to an $\alpha \in R_d$. Removing the walls from $N^1(Y)_\mathbb{R}$ leaves a finite set of open convex cones called chambers. The action of W_d permutes these chambers.

We recall the following theorem about the effective roots on Y, proved in [8]:

Theorem 5.1 (Demazure)**.** *Let Y be a split generalized Del Pezzo surface of degree $d \leq 6$ and let R'_Y be the set of irreducible, effective roots on Y. Then $R_Y := R'_Y \cup -R'_Y$ is a closed symmetric subset of R_d. In particular, R_Y is a root system in the vector space it spans.*

In this section we prove the following theorem:

Theorem 5.2. *Let Y be a split generalized Del Pezzo surface of degree $d \leq 6$. With notation as above, let $W_Y \subset W_d$ be the Weyl group of linear transformations of $N^1(Y)_{\mathbb{R}}$ generated by reflections through roots in R_Y. Let X be a split ordinary Del Pezzo surface of the same degree as Y. Then*

$$\alpha(Y) = \frac{1}{\#(W_Y)}\alpha(X).$$

Recall that the values $\alpha(X)$ were found by Derenthal [9] for $d \leq 7$ and by Peyre [16, Lemme 9.4.2] for $d = 8$ (see Table 1).

Proof. Identify $N^1(X)$ and $N^1(Y)$ as in Remark 3.5. Recall that $\mathrm{Eff}^1(X)$ is the cone generated by the (-1)-classes in $N^1(X)_{\mathbb{R}}$, and $\mathrm{Eff}^1(Y)$ is generated by those same classes together with those irreducible (-2)-classes (roots) which are effective on Y. Hence $\mathrm{Eff}^1(X)$ is contained in $\mathrm{Eff}^1(Y)$ and the reverse inclusion of nef cones follows by duality.

Let C be the open convex cone in $N_1(Y)_{\mathbb{R}}$ dual to the cone spanned by R'_Y. That is, $C = \{\, v \in N_1(Y)_{\mathbb{R}} : \langle v, \alpha\rangle > 0 \text{ for all } \alpha \in R'_Y \,\}$. Since R'_Y is a positive system of roots in the sense of [6], C is a single chamber for the Weyl group W_Y. Furthermore,

$$\mathrm{Nef}(Y) = \overline{C} \cap \mathrm{Nef}(X), \quad \mathcal{P}_Y = \overline{C} \cap \mathcal{P}_X.$$

Indeed, the first equation follows from the dual statement that $\mathrm{Eff}^1(Y)$ is the sum of $\mathrm{Eff}^1(X)$ and the cone generated by R'_Y, which is dual to C. The second equation is obtained by intersecting with the hyperplane $\mathcal{H}_X$.

Then we have the following:

$$N^1(X)_{\mathbb{R}} = \bigcup_{w \in W_Y} \overline{wC},$$

so

$$\mathcal{P}_X = \bigcup_{w \in W_Y} \left(\overline{wC} \cap \mathcal{P}_X\right).$$

The sets $\overline{wC} \cap \mathcal{P}_X$, $w \in W_Y$, are pairwise disjoint except along boundaries, which have zero volume. The action of W_Y preserves volume and fixes $\mathrm{Nef}(X)$ and $-K_X$. Therefore it fixes

$\mathcal{P}_X$, and we have the following.

$$\begin{aligned}\alpha(X) &= \mathrm{Vol}(\mathcal{P}_X)\\ &= \sum_{w\in W_Y} \mathrm{Vol}(\overline{wC}\cap\mathcal{P}_X)\\ &= \#(W_Y)\cdot \mathrm{Vol}(\overline{C}\cap\mathcal{P}_X)\\ &= \#(W_Y)\cdot \mathrm{Vol}(\mathcal{P}_Y)\\ &= \#(W_Y)\cdot \alpha(Y).\end{aligned}$$

This shows $\alpha(Y) = \dfrac{1}{\#(W_Y)}\alpha(X)$. □

Remark 5.3. Theorem 5.2 holds as well for $d = 7$ by essentially the same argument. The set R_7 of roots does not span the hyperplane $(-K_X)^\perp$. Adjusting for this, the proof given above carries through in the case $d = 7$.

6. Non-split Generalized Del Pezzo Surfaces

The previous two sections give two different methods to compute the invariant $\alpha(Y)$ when Y is a split generalized Del Pezzo surface. In this section, we make some elementary observations that are useful in computing α for non-split generalized Del Pezzo surfaces and give some numerical data for high degree cases. Already in these instances it is apparent that the situation is much more complicated than in the split case.

The Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on $N^1(\overline{Y})$, and each automorphism of $N^1(\overline{Y})$ induced by an element of G_K preserves both the intersection form and the anticanonical class.

Proposition 6.1. *Let Y be a generalized Del Pezzo surface of degree $d \geq 6$. Then the group of automorphisms of $N^1(\overline{Y})$ which preserve the intersection form $\langle\cdot,\cdot\rangle$ and the anticanonical class $-K_Y$ is canonically isomorphic (via restriction to $-K_Y^\perp$) to $W(R_d)$.*

Proof. The result for ordinary Del Pezzo surfaces is Theorem 23.9 in [15]. The result holds for generalized Del Pezzo surfaces via the identification described in Remark 3.5. □

Lemma 6.2. *Let Y_1 and Y_2 be two generalized Del Pezzo surfaces defined over a number field K which are geometrically isomorphic, i.e., $\overline{Y_1} \cong \overline{Y_2}$. Let H_1 and H_2 denote the images of G_K under the respective homomorphisms $G_K \to W(R_d)$. If H_1 and H_2 are conjugate in $W(R_d)$, then $\alpha(Y_1) = \alpha(Y_2)$.*

Proof. As in Remark 3.5, identify $N^1(Y_1)$ and $N^1(Y_2)$. Suppose that there is a $w \in W(R_d)$ such that $H_2 = wH_1w^{-1}$. The effective cone of $\overline{Y}$ is generated by the irreducible, effective (-1)- and (-2)-classes according to Theorem 3.1, and these classes are permuted by the Galois action. Let $\mathcal{O}_i$, $i \in I$, denote the orbits of these classes under H_1. Recall that by

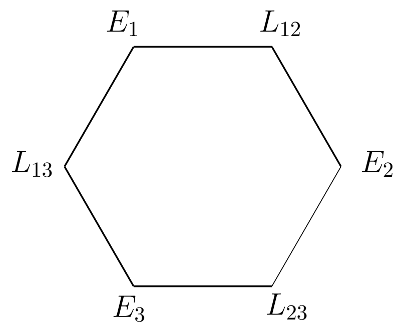

FIGURE 1. Ordinary Del Pezzo surface of degree 6

Corollary 3.7, $\mathrm{Eff}^1(Y_1)$ is generated by the sums $\sum_{D\in\mathcal{O}_i} D$, $i \in I$. A simple calculation shows that the orbits of these classes under H_2 is given by $w\mathcal{O}_i$, $i \in I$. We have

$$\alpha(Y_1) = \mathrm{Vol}\left(\left\{C \in N_1(Y)_{\mathbb{R}} : \langle -K_{Y_1}, C\rangle = 1, \langle C, \sum_{D\in\mathcal{O}_i} D\rangle \geq 0 \ \forall i \in I\right\}\right).$$

Making use of the fact that elements of $W(R_d)$ preserve the intersection form and anticanonical class, identifying $-K_{Y_1}$ and $-K_{Y_2}$, and noting that elements of $W(R_d)$ are orthogonal transformations and thus preserve volumes, we compute

$$\begin{aligned}
\alpha(Y_2) &= \mathrm{Vol}\left(\left\{C \in N_1(Y)_{\mathbb{R}} : \langle -K_{Y_2}, C\rangle = 1, \langle C, \sum_{D\in\mathcal{O}_i} wD\rangle \geq 0 \ \forall i \in I\right\}\right)\\
&= \mathrm{Vol}\left(\left\{C \in N_1(Y)_{\mathbb{R}} : \langle -K_{Y_2}, w^{-1}C\rangle = 1, \langle w^{-1}C, \sum_{D\in\mathcal{O}_i} D\rangle \geq 0 \ \forall i \in I\right\}\right)\\
&= \mathrm{Vol}\left(w\left\{C \in N_1(Y)_{\mathbb{R}} : \langle -K_{Y_1}, C\rangle = 1, \langle C, \sum_{D\in\mathcal{O}_i} D\rangle \geq 0 \ \forall i \in I\right\}\right)\\
&= \alpha(Y_1).
\end{aligned}$$

□

There are very few non-trivial Galois actions possible for generalized Del Pezzo surfaces of degree ≥ 7. Indeed only two cases arise. First, Y can be a twist of $\mathbb{P}^1 \times \mathbb{P}^1$ in which the Galois action permutes the two generating rulings, in which case $\alpha(Y) = \frac{1}{2}$. Second, if $\overline{Y}$ is the blow-up of two conjugate rational points, then $\alpha(Y) = \frac{1}{6}$.

Next we consider the various possible Galois orbits for ordinary Del Pezzo surfaces of degree 6, with the minimal set of generators for $\mathrm{Eff}^1(\overline{Y})$ given as in Figure 1. In this graph, the vertices correspond to the generating classes for $\mathrm{Eff}^1(\overline{Y})$, with the convenient shorthand $L_{ij} = L - E_i - E_j$. Two classes intersect if and only if the corresponding vertices in the graph are connected by an edge.

Table 3. Ordinary Del Pezzo surfaces of degree 6

Orbit structure of Y	ρ	m	$\alpha(Y)$
$\{E_1, E_2, E_3, L_{12}, L_{13}, L_{23}\}$	1	1	1
$\{E_1, E_2, E_3\}, \{L_{12}, L_{13}, L_{23}\}$	2	2	1
$\{E_1, L_{12}\}, \{E_2, L_{13}\}, \{E_3, L_{23}\}$	2	2	$\frac{1}{2}$
$\{E_1, L_{23}\}, \{E_2, L_{13}\}, \{E_3, L_{12}\}$	3	3	$\frac{1}{2}$
$\{E_1, E_3\}, \{E_2\}, \{L_{13}\}, \{L_{12}, L_{23}\}$	3	4	$\frac{1}{12}$

Table 4. Generalized Del Pezzo surfaces of degree 6

Deformation class of Y	ρ	m	$\alpha(Y)$
$\mathbf{A}_1$ (4 lines)	3	3	$\frac{1}{8}$
$\mathbf{A}_1$ (3 lines)	3	3	$\frac{1}{24}$
$\mathbf{A}_1$ (3 lines)	2	2	$\frac{1}{6}$
$\mathbf{A}_2$	3	3	$\frac{1}{48}$

In Table 3, we consider the possible orbit structures of subgroups of $W(R_6) = \mathbb{Z}/2\mathbb{Z} \times \mathcal{S}_3$. For each orbit structure (given up to the action of $W(R_6)$), we give the rank ρ of $N^1(Y)$, the number m of generators of $\operatorname{Eff}^1(Y)$, and finally the invariant $\alpha(Y)$.

In the case of generalized Del Pezzo surfaces of degree 6, there are relatively few possible non-trivial Galois actions. This can be seen by looking at the analogous Dynkin graphs whose vertices are the minimal generators for $\operatorname{Eff}^1(\overline{Y})$ and whose edges correspond to intersecting classes. These graphs are given in [7], so we do not reproduce them here. It turns out that the various possible orbit structures can be uniquely determined—up to the action of $W(R_6)$—by specifying the deformation class of Y along with the number of Galois orbits, which is simply the number m of generators of $\operatorname{Eff}^1(Y)$.

In the case of ordinary Del Pezzo surfaces of degree 5, we use a different diagram to keep track of the minimal generators, because it is difficult to clearly draw the ordinary Dynkin diagram in two dimensions that illustrates its full symmetry under $W(R_5) = \mathcal{S}_5$. Instead, we use the graph shown in Figure 2. Here the minimal generators of $\operatorname{Eff}^1(\overline{Y})$ correspond to *edges* of the graph, and two generating classes intersect if and only if they do not share a common vertex. We use the notation $\overline{ij}$ to indicate the edge connecting vertex i with vertex j. The action of $W(R_5) = \mathcal{S}_5$ corresponds to permuting the 5 vertices. One can use Table 5 to identify edges with the common representation of the generating classes in the standard $L, E_1, \ldots, E_4$ basis.

In order to enumerate the possible Galois orbits, we first enumerate the conjugacy classes of subgroups of $\mathcal{S}_5$. This has already been done by Götz Pfeiffer and is available online [17]. We use his numbering of the subgroups to identify them in Table 6. (Note: We omit the first conjugacy class in Pfeiffer's table, which is the trivial group and corresponds to the split case already discussed.)

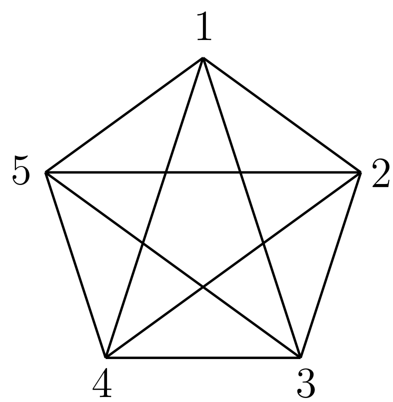

FIGURE 2. Ordinary Del Pezzo surface of degree 5

TABLE 5. Correspondence of edges in Figure 2 to generators of the effective cone of an ordinary Del Pezzo surface of degree 5

$\overline{12}$	$\overline{13}$	$\overline{14}$	$\overline{15}$	$\overline{23}$	$\overline{24}$	$\overline{25}$	$\overline{34}$	$\overline{35}$	$\overline{45}$
E_1	E_2	E_3	E_4	L_{34}	L_{24}	L_{23}	L_{14}	L_{13}	L_{12}

TABLE 6. Ordinary Del Pezzo surfaces of degree 5

Orbit structure of Y	Conj. classes	ρ	m	$\alpha(Y)$
$\{E_1, E_2, E_3, E_4, L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}\}$	13,16,18,19	1	1	1
$\{E_1, E_2, E_3, L_{14}, L_{24}, L_{34}\}, \{E_4, L_{12}, L_{13}, L_{23}\}$	12,14,17	2	2	$\frac{4}{3}$
$E_1, \{E_2, E_3, E_4, L_{23}, L_{24}, L_{34}\}\{L_{12}, L_{13}, L_{14}\}$	10,11,15	2	2	$\frac{1}{2}$
$\{E_1, E_2, L_{34}\}, \{E_3, L_{14}, L_{24}\}, \{E_4, L_{13}, L_{23}\}, L_{12}$	4, 9	3	4	$\frac{5}{8}$
$\{E_1, E_4, L_{12}, L_{14}, L_{34}\}, \{E_2, E_3, L_{13}, L_{23}, L_{24}\}$	8	1	1	1
$\{E_1, L_{14}\}, \{E_2, L_{24}\}, \{E_3, L_{34}\}, \{E_4, L_{12}, L_{13}, L_{23}\}$	7	2	2	$\frac{4}{3}$
$E_1, \{E_2, E_3, L_{24}, L_{34}\}, \{E_4, L_{23}\}, \{L_{12}, L_{13}\}, L_{14}$	6	3	4	$\frac{2}{9}$
$\{E_1, E_3, L_{14}, L_{34}\}, \{E_2, L_{24}\}, \{E_4, L_{12}, L_{13}, L_{23}\}$	5	2	2	$\frac{4}{3}$
$E_1, \{E_2, L_{34}\}, \{E_3, L_{24}\}, \{E_4 L_{23}\}, \{L_{12}, L_{13}\}, L_{14}$	3	3	4	$\frac{2}{9}$
$E_1, \{E_2, L_{34}\}, \{E_3, L_{24}\}, \{E_4 L_{23}\}, L_{14}, L_{13}, L_{12}$	2	4	5	$\frac{5}{72}$

As with generalized Del Pezzo surfaces of degree 6, we can handle generalized Del Pezzo surfaces of degree 5 by enumerating possible Galois actions using the diagrams given in [7] and noting again that deformation type and the number m of Galois orbits uniquely determines the orbit structure, up to the action of the Weyl group $W(R_5)$. The results are given in Table 7.

It is an open problem whether there is a systematic method for computing $\alpha(Y)$ for all non-split generalized Del Pezzo surfaces.

Table 7. Generalized Del Pezzo surfaces of degree 5

Deformation class of Y	ρ	m	$\alpha(Y)$
$\mathbf{A}_1$	4	6	$\frac{5}{432}$
$\mathbf{A}_1$	3	4	$\frac{5}{48}$
$2\mathbf{A}_1$	3	4	$\frac{1}{12}$
$\mathbf{A}_2$	4	5	$\frac{5}{144}$

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